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ex. 1) a) Prove that we get a topology for $\mathbb{N}$ by taking
$\mathcal{T} = \{\phi, \mathbb{N}, \{1,2,\dots,n\} : n \in \mathbb{N}\}$

Proof:

(T1) We have $\phi \in \mathcal{T}$ and $\mathbb{N} \in \mathcal{T}$, by how $\mathcal{T}$ is defined

(T2) Let $U, V \in \mathcal{T}$
Then we have to proof $U \cap V \in \mathcal{T}$
If U or V is the empty set, we have $U \cap V = \phi \in \mathcal{T}$ (same if $U = \phi$ and $V = \phi$)
If $\{U,V\} = \{\mathbb{N}, \{1,2,\dots,n\}\}$ for some $n \in \mathbb{N}\}$,
then $U \cap V = \{1,2,\dots,n\} \in \mathcal{T}$
If $U = \mathbb{N} = V$, then $U \cap V = \mathbb{N} \in \mathcal{T}$
If $U = \{1,2,\dots,n\}$ and $V = \{1,2,\dots,m\}$ for some $n, m \in \mathbb{N}$,
~~then~~ and suppose without loss of generality that $n \geq m$
then $U \cap V = V \in \mathcal{T}$
So for any $U, V \in \mathcal{T}$, $U \cap V \in \mathcal{T}$

(T3) Let $U_1, U_2, \dots$ be any collection of open sets,
then we have to proof $\bigcup_{i \in I} U_i$ is open in $\mathbb{N}$.

If ~~any~~ $U_i = \mathbb{N}$ for some $i \in I$, then
$\bigcup_{i \in I} U_i = \mathbb{N} \in \mathcal{T}$.
So suppose $U_i \neq \mathbb{N}$ for any $i \in I$.
Then $\bigcup_{i \in I} U_i = \{1,2,\dots,N\} \in \mathcal{T}$ where $N = \max\{n_i : i \in I\}$ if the ~~each~~ n_i ~~is finite (~~ are bounded above
and $\bigcup_{i \in I} U_i = \mathbb{N} \in \mathcal{T}$ if $n_i \to \infty$
If $U_i = \phi$ for all $i \in I$, then $\bigcup_{i \in I} U_i = \phi \in \mathcal{T}$
So any collection of open sets is again open.

So $\mathcal{T}$ is a topology for $\mathbb{N}$ $\square$

b) Prove that $(\mathbb{N}, \mathcal{T})$ is not compact and is not Hausdorff.

Proof: Let $\mathcal{U} = \{U_n = \{1, 2, \dots, n\} : n = 1, 2, \dots\}$

Then $\mathcal{U}$ is an open cover for $\mathbb{N}$

now suppose $(\mathbb{N}, \mathcal{T})$ is compact, then

$\mathcal{U}$ has a finite subcover $\mathcal{U}' = \{U_{n_1}, \dots, U_{n_k}\}$

Let $N = \max\{n_1, \dots, n_k\}$, ~~then~~

~~then $\{1, 2, \dots, N+1\} \notin \mathcal{U}'$, while~~

then $N+1 \notin U_{n_i}$ for $i = 1, \dots, k$, so

$\mathcal{U}'$ is not an open cover for $\mathbb{N}$, thus $\mathcal{U}$ has no finite subcover. So $(\mathbb{N}, \mathcal{T})$ is not compact.

now to proof $\{\mathbb{N}, \mathcal{T}\}$ is not Hausdorff.

Let $1, n \in \mathbb{N}$ for any n, then for any 2 subsets U, V ~~with~~ with $n \in V$, $1 \in U$ we have $U \cap V \neq \emptyset$, since $1 \in U$ and $1 \in V$ as well by how the topology is defined.

So there exist no open sets ~~such that~~ U, V such that $1 \in U$, $n \in V$ and $U \cap V = \emptyset$

Hence $(\mathbb{N}, \mathcal{T})$ is not Hausdorff. $\square$

ex. 2) Suppose (X, d) is a metric space and consider a map $f : X \to X$.

a) Show that for all $x, y \in X$

$$|d(f(x), x) - d(f(y), y)| \leq d(f(x), f(y)) + d(x, y)$$

Proof: Using the triangle inequality we get

$d(f(x), x) \leq d(f(x), y) + d(x, y)$ and

$d(f(x), y) \leq d(f(x), f(y)) + d(f(y), y)$

So $d(f(x), x) \leq d(f(x), f(y)) + d(f(y), y) + d(x, y)$

or $d(f(x), x) - d(f(y), y) \leq d(f(x), f(y)) + d(x, y)$

Similarly we get

$d(f(y),y) \leq d(f(y),x) + d(x,y) \leq d(f(y),f(x)) + d(f(x),x) + d(x,y)$

so $d(f(y),y) - d(f(x),x) \leq d(f(x),f(y)) + d(x,y)$

And hence we get $|d(f(x),x) - d(f(y),y)| \leq d(f(x),f(y)) + d(x,y)$ $\square$

b) Suppose that $f: X \to X$ is continuous. Then prove that $g: X \to \mathbb{R}$ given by $g(x) = d(f(x),x)$ is also continuous.

Proof: $f: X \to X$ is continuous, so for all $\varepsilon > 0$ and all $x, y \in X$ there is $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

For any $\varepsilon > 0$ and all $x, y \in X$ we need to find $\delta > 0$ such that $d(x,y) < \delta$ implies $|g(x) - g(y)| < \varepsilon$.

$|g(x) - g(y)| = |d(f(x),x) - d(f(y),y)| \leq d(f(x), f(y)) + d(x,y)$

From continuity of f we can find $\delta_0 > 0$ such that when $d(x,y) < \delta_0$ we have $d(f(x), f(y)) < \frac{\varepsilon}{2}$

Now let $\delta = \min\{\delta_0, \frac{\varepsilon}{2}\}$, then for all $x, y \in X$ with $d(x,y) < \delta$ we have $|g(x) - g(y)| \leq d(f(x),f(y)) + d(x,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence g is continuous at all points in X, thus g is continuous. $\square$

c) Further suppose X is compact and that $f(x) \neq x$ for all $x \in X$. Then show that there is $\varepsilon > 0$ such that $d(f(x),x) \geq \varepsilon$ for all $x \in X$.

Proof: Since $f(x) \neq x$ for all $x \in X$ we know that $d(f(x),x) \neq 0$ for all $x \in X$.

And since d is a metric $d(f(x),x) \geq 0$ for all $x \in X$

Since g is continuous and X is compact, we know g attains its bounds in X.

So there is a $c \in X$ with $g(c) = \inf \{g(x) : x \in X\}$

and since $g(x) > 0$ for all $x \in X$, we have

$g(c) > 0$

So there is $0 < \varepsilon$ such that $0 < \varepsilon \leq g(c)$

But since $g(c) = \inf \{g(x) : x \in X\}$

we have $\varepsilon \leq g(c) \leq g(x) = d(f(x), x)$ for all $x \in X$ □

ex. 3)

a) Suppose that $f: X \to Y$ is a surjective continuous map from a path-connected topological space X to a topological space Y. Show that Y is path-connected

Proof: Take any $y_1, y_2 \in Y$. Then since f is surjective there $x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$.

Since X is path-connected there is a ~~map~~ continuous map $h: [0,1] \to X$ with $h(0) = x_1$ and $h(1) = x_2$.

Then $f \circ h : [0,1] \to Y$ is continuous as the composition of two continuous maps, and $f \circ h(0) = f(h(0)) = f(x_1) = y_1$ and

$f \circ h(1) = f(h(1)) = f(x_2) = y_2$

So for any $y_1, y_2 \in Y$ there is a continuous map $f \circ h : [0,1] \to Y$ with $f \circ h(0) = y_1$ and $f \circ h(1) = y_2$

So Y is path-connected □

b) Prove that the set $A = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 2\}$ is path-connected

Proof: We define $f : \mathbb{R}^2 \to A$ by almost! but good enough

$$f(x,y) = ((1+\sin^2 y)\cos x, (1+\sin^2 y)\sin x)$$

This map is well defined since $1 \le 1+\sin^2 y \le 2$ for all $y \in \mathbb{R}$, ~~and that~~ ~~continuous~~ and $(\cos x, \sin y)$ ~~are~~ is a points on the unit circle.

f is surjective, since for $(x_1,y_1) \in A$ we have $1 \le x_1^2+y_1^2 \le 2$, then

$(x_1,y_1) = (R^2\cos\varphi, R^2\sin\varphi)$ for $R^2 = x_1^2+y_1^2$ and $\varphi \in [0,2\pi]$

So let $(1+\sin^2 y) = R$ and $x = \varphi$

and we get $f(x,y) = ((1+\sin^2 y)\cos x, (1+\sin^2 y)\sin x) = (R\cos\varphi, R\sin\varphi) = (x_1,y_1)$

Now we want to show f is continuous

We know f is continuous iff $i \circ f$ is continuous

But $i\circ f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous as the product of continuous functions

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[dr, "i\circ f"'] & A \arrow[d, "i"] \\ & \mathbb{R}^2 \end{tikzcd}$$

$g : \mathbb{R} \to \mathbb{R}^2$, $g(x) = (\cos x, \sin x)$

$h : \mathbb{R} \to \mathbb{R}^2$, $h(y) = (1+\sin^2 y, 1+\sin^2 y)$

So f is continuous

We also know $\mathbb{R}^2$ is path-connected.

So we have a surjective, continuous map f from a path-connected topological space $\mathbb{R}^2$ to A.

So from (a) A is path-connected.

ex. 4) Given a topological space X, the diagonal subset Δ of $X \times X$ is defined as $\Delta = \{(x,x) \in X \times X : x \in X\}$.
Prove that ~~if~~ X is Hausdorff iff Δ is closed in the topological product $X \times X$

<u>Proof:</u> Suppose X is Hausdorff, then we need to proof Δ is closed in $X \times X$ or equivalently to proof $X \times X \setminus \Delta$ is open in $X \times X$.
$X \times X \setminus \Delta$ is open iff for all $(x,y) \in X \times X \setminus \Delta$ there are sets $U_x, V_y \in \mathcal{T}_X$ with $x \in U_x$, $y \in V_y$ and $(U,V) \subseteq X \times X \setminus \Delta$
Take any two points $(x,y) \in X \times X \setminus \Delta$, then we know $x \neq y$. Since X is Hausdorff, there are open sets $U_x, V_y \in \mathcal{T}_X$ with $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \emptyset$.
Since $U_x \cap V_y = \emptyset$, we have $x \notin V_y$ and $y \notin U_x$.
So for any $(x_1, y_1) \in (U_x, V_y)$ we have $x_1 \neq y_1$, so $(U_x, V_y) \subseteq X \times X \setminus \Delta$.
$U_x \times V_y$
So we have $U_x, V_y \in \mathcal{T}_X$ with $(x,y) \in (U_x, V_y) \subseteq X \times X \setminus \Delta$ and thus $X \times X \setminus \Delta$ is open in $X \times X$, and hence Δ is closed in $X \times X$.

Now suppose Δ is closed in $X \times X$, then $X \times X \setminus \Delta$ is open.
We need to proof X is Hausdorff.
Since $X \times X \setminus \Delta$ is open, for any $(x,y) \in X \times X \setminus \Delta$ there are ~~ope~~ sets $U_x, V_y \in \mathcal{T}_X$ with $(x,y) \in (U_x, V_y) \subseteq X \times X \setminus \Delta$.
Now $(U_x, V_y) \subseteq X \times X \setminus \Delta$ implies that for all $x_1 \in U_x$, $y_1 \in V_y$ we have $x_1 \neq y_1$ and hence $U_x \cap V_y = \emptyset$
So for any $x, y \in X$ with $x \neq y$ we found open sets U_x, V_y such that $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \emptyset$.
Hence X is Hausdorff $\square$